2/04/2014 | Lec. 3.

If $\sum_{n=1}^{\infty} a_n$ conv., then $\lim_{n\to\infty} a_n = 0$.

Converse **not** true.

E.g., $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$.

What about $\sum_{n=1}^{\infty} \frac{1}{n^2}$?

In fact, this converges.

Lemma. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

Proof. Note that $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$.

$S_1 = \frac{1}{1(1+1)} = \frac{1}{2}$.

$S_2 = \frac{1}{2} + \frac{1}{2\cdot 3} = \frac{1}{1\cdot 2} + \frac{1}{2} - \frac{1}{3}$,

$S_3 = \frac{1}{2} + \underbrace{\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}} + \underbrace{\cancel{\frac{1}{3}} - \frac{1}{4}} = \frac{1}{2} + \frac{1}{2} - \frac{1}{4}$,

$S_4 = \frac{1}{2} + \underbrace{\frac{1}{2} - \cancel{\frac{1}{3}}} + \underbrace{\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}} + \underbrace{\cancel{\frac{1}{4}} - \frac{1}{5}} = 1 - \frac{1}{5}$,

$S_5 = \frac{1}{2} + \underbrace{\frac{1}{2} - \cancel{\frac{1}{3}}} + \underbrace{\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}} + \underbrace{\cancel{\frac{1}{4}} - \cancel{\frac{1}{5}}} + \underbrace{\cancel{\frac{1}{5}} - \frac{1}{6}} = 1 - \frac{1}{6}$

This suggests

$S_n = 1 - \frac{1}{n+1}$. (You could verify this by induction).

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1 . \qquad \square$$

Because $\frac{1}{n^2} > \frac{1}{n(n+1)}$, we're done

<u>NO!</u>

Notice that $\cancel{\frac{1}{n^2} < \frac{1}{(n+1)(n+2)}}$

$$\frac{1}{(n+1)^2} < \frac{1}{n(n+1)} .$$

But
$$\underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2}}_{= 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots} = \underbrace{\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}}_{= \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots} + 1 .$$

Now, $\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$ converges by Lemma and comparison. Thus, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

In fact, $\sum_{n=1}^{\infty} \frac{1}{n^2} \in (1, 2)$.

By comparison,

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges when $p \geq 2$.

Recall that it diverges for $p \in (0, 1]$.

What about $p \in (\overset{1}{\cancel{0}}, 2)$?

We don't know, but we will.

What about $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$?

$" -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots$

Actually, this converges.

<u>Theorem.</u> If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\cancel{\sum_{n=1}^{\infty} |a_n|}$ $\sum_{n=1}^{\infty} a_n$ converges.

<u>Note</u> Converse does <u>not</u> hold.

<u>Proof.</u> Observe that

$$0 \leq a_n + |a_n| \leq 2|a_n|.$$

$\sum_{n=1}^{\infty} 2|a_n|$ converges because

$\sum_{n=1}^{\infty} |a_n|$ conv. Therefore,

$\sum_{n=1}^{\infty} (a_n + |a_n|)$ conv. by comparison.

Now, $a_n = \underbrace{(a_n + |a_n|)}_{\text{conv.}} - \underbrace{|a_n|}_{\text{conv.}}$

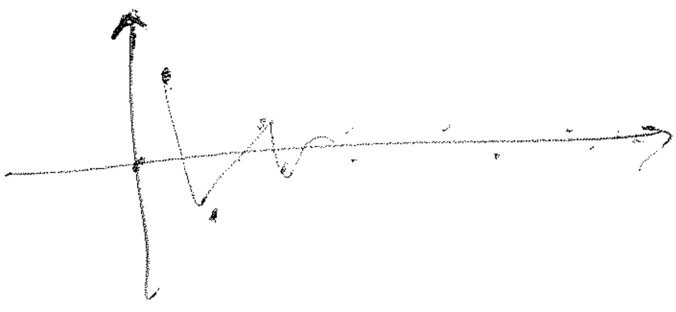
Therefore, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (a_n + |a_n|)$

$$- \sum_{n=1}^{\infty} |a_n|,$$

converges. $\square$

<u>Theorem</u> (Leibniz test). If $(a_n)_{n=1}^{\infty}$ is a nonnegative, nonincreasing seq. and $\lim_{n\to\infty} a_n = 0$, then

$\sum_{n=1}^{\infty} (-1)^n a_n$ converges.



This implies

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ conv.

Def. If $\sum_{n=1}^{\infty} |a_n|$ converges,

we say $\sum_{n=1}^{\infty} a_n$ converges absolutely.

If $\sum_{n=1}^{\infty} a_n$ converges, but

$\sum_{n=1}^{\infty} |a_n|$ diverges, then $\sum_{n=1}^{\infty} a_n$ converges conditionally. (?)

Theorem (ratio test, Cauchy test).

Assume $a_n \neq 0$.

1) If $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, then $\sum_{n=1}^{\infty} a_n$ conv.

2) If $\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$, then it diverges.

3) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, we don't know. (test is inconclusive)

Theorem (root test, D'Alamert (?)).

Set $\rho = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$.

1) If $\rho < 1$, $\sum_{n=1}^{\infty} a_n$ converges.

2) If $\rho > 1$, it diverges.

3) If $\rho = 1$, test doesn't apply.

Lec 14 |

EX. Consider the series $\sum_{n=1}^{\infty} a_n$, where $a_n = \begin{cases} \frac{1}{2^n}, & n \text{ odd}, \\ \frac{1}{3^n}, & n \text{ even}. \end{cases}$

$$\frac{1}{2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^4} + \frac{1}{2^5} + \cdots$$

Ratio test.

$$\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \limsup_{n\to\infty} \begin{cases} \frac{1}{2^{n+1}} \Big/ \frac{1}{3^n}, & n \text{ even}, \\ \frac{1}{3^{n+1}} \Big/ \frac{1}{2^n}, & n \text{ odd}, \end{cases}$$

$$= \limsup_{n\to\infty} \begin{cases} \left(\frac{3}{2}\right)^n \cdot \frac{1}{2}, & n \text{ even}, \\ \left(\frac{2}{3}\right)^n \cdot \frac{1}{3}, & n \text{ odd}. \end{cases}$$

$$= \infty. \quad (>1)$$

$$\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0 \, (<1).$$

Thus, ratio test doesn't give us any info.

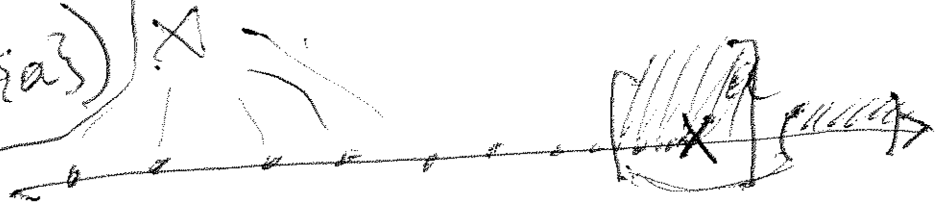
Try root test.

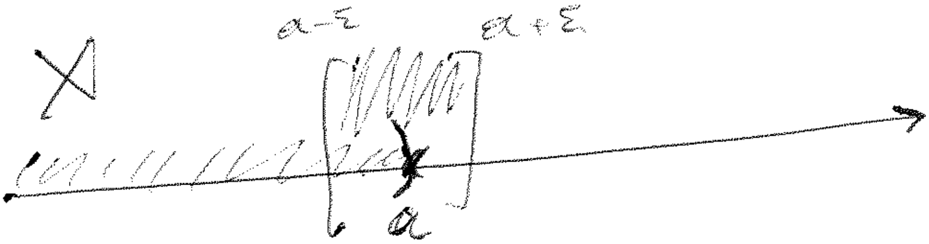
We compute

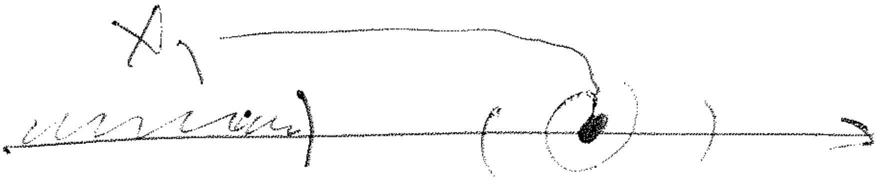
$$\sqrt[n]{|a_n|} = \begin{cases} \sqrt[n]{\frac{1}{3^n}}, & n \text{ even}, \\ \sqrt[n]{\frac{1}{2^n}}, & n \text{ odd}, \end{cases} = \begin{cases} \frac{1}{3}, & n \text{ even}, \\ \frac{1}{2}, & n \text{ odd}. \end{cases}$$

$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \frac{1}{2} < 1$. Thus, the series converges. $\square$

<u>Limit point.</u> Assume X is a subset of $\mathbb{R}$. The point $a \in \mathbb{R}$ is a <u>limit point</u> of X if <u>for every $\varepsilon > 0$ there exist</u> $x \in X$ s.t. $x \in \big((a-\varepsilon, a+\varepsilon) \setminus \{a\}\big)$

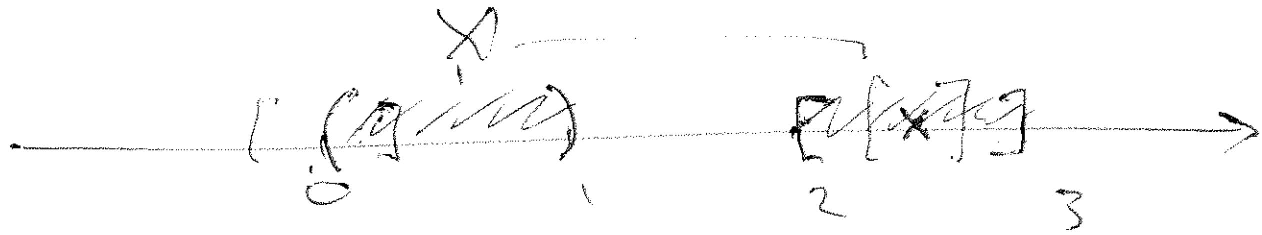






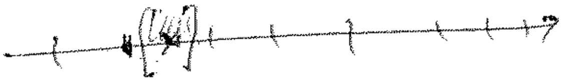
<u>Note.</u> a need not be in X.

Examples. 1) $X = (0,1) \cup [2,3]$.



Limit points of X are $[0,1] \cup [2,3]$.

2) $\mathbb{Z}$ has no limit points.

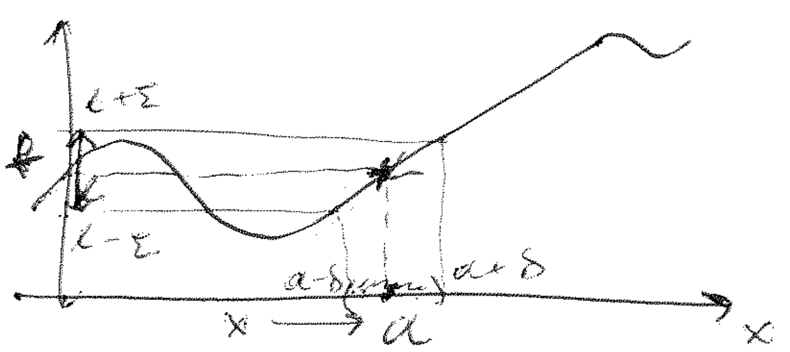


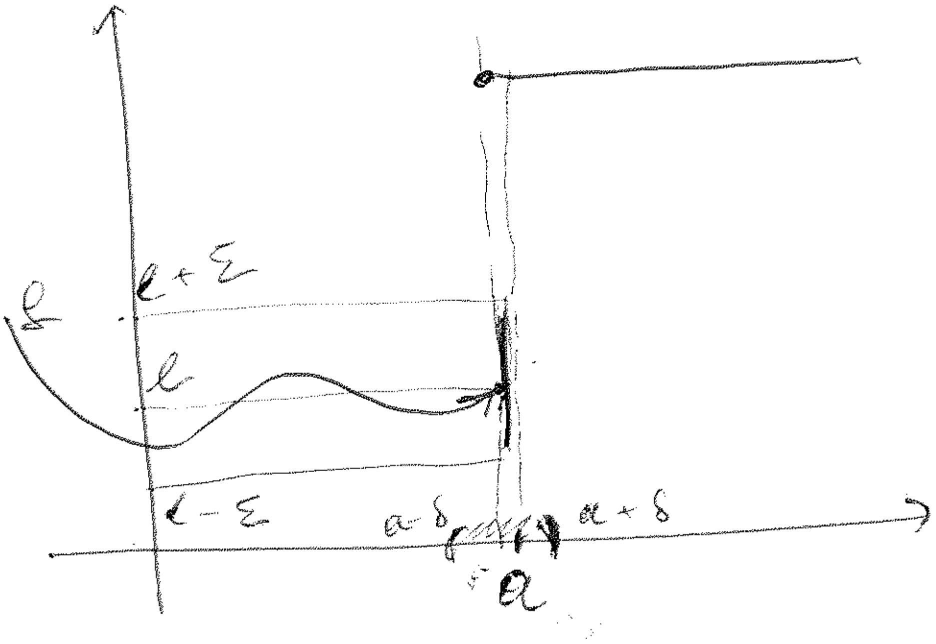
3) $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\}$

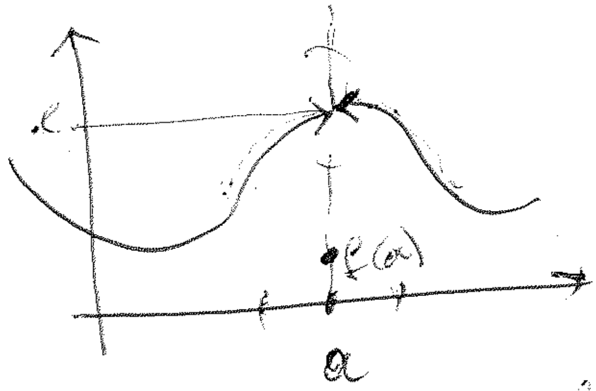
Limit points: $\{0\}$.

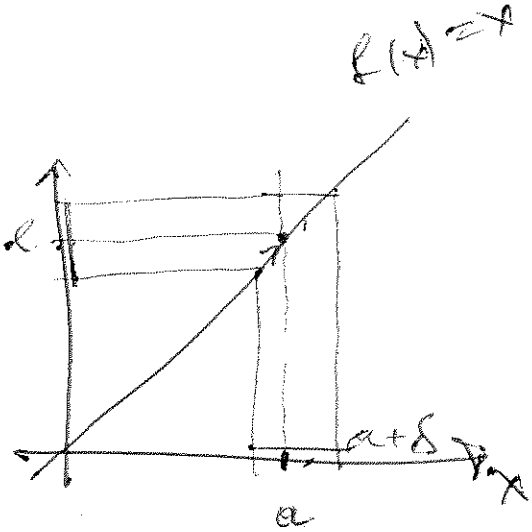
Def. Consider a function $f: X \to \mathbb{R}$. Assume a is a limit point of X.

$\lim_{x \to a} f(x) = \ell$ if for every $\varepsilon > 0$ there exists $\delta > 0$ s.t. if $0 < |x-a| < \delta$, then $|f(x) - \ell| < \varepsilon$.









Example. 1) $\lim\limits_{x\to a} \overset{=f(x)}{x} = \overset{=l}{a}$.

Take $\varepsilon > 0$. Assume $\delta = \varepsilon$.

If $0 < |a - x| < \delta$, then

$|\underset{l}{\underset{\shortparallel}{a}} - \underset{f(x)}{\underset{\shortparallel}{x}}| < \varepsilon$. Thus, $\lim\limits_{x\to a} x = a$.

Example 2). $\lim_{x \to 2} (x^2 + x) = 6$.

Proof. Fix $\varepsilon > 0$. We want to find $\delta > 0$ s.t.

$$0 < |x-2| < \delta \Rightarrow |\underbrace{x^2+x}_{f(x)} - \underbrace{6}_{l}| < \varepsilon.$$

Note that

$$|x^2+x-6| = \overbrace{|x-2|}^{<\delta}\,\overbrace{|x+3|}^{\leq 6}.$$

Assume $|x-2| < 1$. Then $|x| \leq 1+2 = 3$. Then

$$|x+3| \leq |x| + 3 \leq 6.$$

Now, take $\delta = \min\{\frac{\varepsilon}{6}, 1\}$.

Then

$$|x^2+x-6| = |x-2||x+3| \leq \delta \cdot |x+3|$$

$$\leq \frac{\varepsilon}{6} \cdot 6 = \varepsilon.$$

$\square$